

On elliptic ovoids and their rosettes in a classical generalized quadrangle of even order.

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Abstract

Let Q_0 be the classical generalized quadrangle of order $q = 2^n$ arising from a non-degenerate quadratic form in a 5-dimensional vector space defined over a finite field of order q . We consider the rank two geometry \mathcal{X} having as points all the elliptic ovoids of Q_0 and as lines the maximal pencils of elliptic ovoids of Q_0 pairwise tangent at the same point. We first prove that \mathcal{X} is isomorphic to a 2-fold quotient of the affine generalized quadrangle $Q \setminus Q_0$ where Q is the classical (q, q^2) -generalized quadrangle admitting Q_0 as a hyperplane. Then, we investigate the collinearity graph Γ of \mathcal{X} . In particular, we obtain a classification of the cliques of Γ proving that they arise either from lines of Q or subgeometries of Q defined over \mathbb{F}_2 .

1 Introduction

We refer to the monograph [12] for the terminology and basics on finite generalized quadrangles.

We only recall that a finite generalized quadrangle \mathcal{Q} is *classical* if its points and lines are points and lines of a projective space $\text{PG}(d, q)$, the points of \mathcal{Q} span $\text{PG}(d, q)$ and all the lines of \mathcal{Q} are full, namely if a projective line l is a line of \mathcal{Q} then all points of l belong to \mathcal{Q} . If this is the case then the points and the lines of \mathcal{Q} are the points and lines of $\text{PG}(d, q)$ that are singular and respectively totally singular for a suitable non-degenerate alternating, hermitian or quadratic form of the underlying vector space $V(d+1, q)$ of $\text{PG}(d, q)$ ([12, Chapter 4]).

The generalized quadrangle of order q arising from a non-degenerate symplectic form on $V(4, q)$ is denoted by $W(q)$. The generalized quadrangle of order q arising from a non-degenerate quadratic form in $V(5, q)$ is denoted by $Q(4, q)$. The (q, q^2) -generalized quadrangle arising from a non-degenerate quadratic form of Witt index 2 in $V(6, q)$ is denoted by $Q^-(5, q)$. The $(q, 1)$ -generalized quadrangle arising from a non-degenerate quadratic form of Witt index 2 in $V(4, q)$ is denoted by $Q^+(3, q)$. It is also called a *grid*. For any q , the dual of $Q(4, q)$ is isomorphic to $W(q)$. Moreover, $Q(4, q) \cong W(q)$ if and only if q is even.

An *ovoid* of an (s, t) -generalized quadrangle \mathcal{Q} is a set X of points of \mathcal{Q} such that every line of \mathcal{Q} meets X in exactly one point (see [12]). Clearly,

$|X| = st + 1$ and no two points of X are collinear. For a survey of ovoids of generalized quadrangles we refer the reader to [11].

In this paper we assume $q = 2^n$ and we denote by Q_0 a classical generalized quadrangle $Q(4, q)$ of order q embedded in a 4-dimensional projective space $\text{PG}(V_0)$, $V_0 = V(5, q)$. Let n_0 be the *nucleus* of Q_0 , i.e. n_0 is the unique non-singular point of $\text{PG}(V_0)$ with the property that the lines of $\text{PG}(V_0)$ through n_0 are the tangents to Q_0 in $\text{PG}(V_0)$.

Classification of ovoids in Q_0 is a fundamental open problem. There are two known classes of ovoids of Q_0 , namely the class \mathcal{E} of *elliptic ovoids* and the class \mathcal{T} of *Tits-Suzuki ovoids*. An ovoid X of Q_0 is elliptic if and only if it spans a hyperplane $\langle X \rangle$ of $\text{PG}(V_0)$ (whence $X = \langle X \rangle \cap Q_0$). Tits-Suzuki ovoids exist if and only if n is odd.

In this paper we will be concerned with a semipartial geometry arising from the class \mathcal{E} of elliptic ovoids of Q_0 . Remind that any two distinct elliptic ovoids of Q_0 intersect in either a singleton or a non-degenerate conic. If two elliptic ovoids meet at a single point p we will say that they are *tangent* at p .

Following [3], we define a *rosette* ρ_p of elliptic ovoids *based* at a point p as a set of q elliptic ovoids of Q_0 mutually intersecting at p such that $\{X \setminus \{p\} : X \in \rho_p\}$ is a partition of the set of the points of Q_0 not collinear with p . The point p is called the *base point* of ρ_p . Clearly, a set of elliptic ovoids mutually tangent at a given point is a rosette if and only if it consists of exactly q ovoids. It is also clear that a rosette with base point p is a maximal pencil of elliptic ovoids of Q_0 mutually tangent at p .

A proof of the following proposition has been given in [1, Theorem 3.2] (see also [6, Example 1.4(d)]).

Proposition 1.1 *Let X_1 and X_2 be two elliptic ovoids of Q_0 tangent at a point p . Then there exists a unique rosette ρ_p with base point p containing X_1 and X_2 .*

Proposition 1.2 *Let ρ_p be a rosette of elliptic ovoids of Q_0 based at the point p . Then there exists a unique projective plane π_p of $\text{PG}(V_0)$ containing p and tangent to each $X \in \rho_p$.*

Proof. Let X_i, X_j, X_k be three distinct elliptic ovoids contained in the rosette ρ_p . Let $\pi_j := \langle X_i \rangle \cap \langle X_j \rangle$ and $\pi_k := \langle X_i \rangle \cap \langle X_k \rangle$ be the intersection planes of the 3-dimensional subspace $\langle X_i \rangle$ spanned by X_i with the 3-dimensional subspaces spanned by X_j and X_k respectively. Since $\langle X \rangle \cap Q_0 = X$ for every $X \in \mathcal{E}$, we have $\pi_j \cap X_i = p = \pi_k \cap X_i$.

If $\pi_j \neq \pi_k$ then π_j and π_k are two distinct planes of $\langle X_i \rangle$ both tangent to X_i at p . This is clearly impossible. Hence, $\pi_j = \pi_k$ and the proposition follows. \square

Let $\mathcal{X} := (\mathcal{E}, \mathcal{L})$ be the point-line geometry having the set \mathcal{E} of elliptic ovoids of Q_0 as the pointset and the set of rosettes of elliptic ovoids of Q_0 as the lineset. An ovoid X is incident with a rosette ρ_p if $X \in \rho_p$. It is known that \mathcal{X} is a semi-partial geometry with parameters $(s, t, \alpha, \mu) = (q - 1, q^2, 2, 2q(q - 1))$

(see [6] and [5]). A very straightforward proof of this claim will also be given in Section 2.

1.1 Main results

Let $Q \cong Q^-(5, q)$ be a classical (q, q^2) -generalized quadrangle embedded in the 5-dimensional projective space $\text{PG}(V)$, where $V = V(6, q)$. We may assume that $Q_0 = Q \cap H_0$ for a suitable hyperplane H_0 of $\text{PG}(V)$. Thus, Q_0 is embedded in $H_0 \cong \text{PG}(4, q)$. Let \perp denote the orthogonality relation defined by Q in $\text{PG}(V)$. The point $n_0 := H_0^\perp$ is the nucleus of the quadric Q_0 .

Let $\hat{\mathcal{X}} := (\hat{P}, \hat{L})$ be the point-line geometry having the set \hat{P} of the points of $Q \setminus Q_0$ as the point-set and the set $\hat{L} := \{l \setminus Q_0 : l \text{ line of } Q \text{ not contained in } Q_0\}$ as line-set.

If l is a line of Q not contained in Q_0 then $l \cap Q_0$ is a point, henceforth denoted by l^∞ . We denote by \hat{l} the line of $\hat{\mathcal{X}}$ corresponding to l and we call l^∞ the *point at infinity* of \hat{l} . Thus,

$$\hat{l} = l \setminus \{l^\infty\}, \text{ and } l = \hat{l} \cup \{l^\infty\}. \quad (1)$$

The incidence relation in $\hat{\mathcal{X}}$ is inherited from the incidence relation of Q . It is easy to see that the incidence graph of $\hat{\mathcal{X}}$ is connected. It has girth 8, whence $\hat{\mathcal{X}}$ has gonality 4. The collinearity graph of $\hat{\mathcal{X}}$ has diameter 3 (see e.g. [10, 8.4.1]).

We recall that a *morphism* of rank 2 geometries is a morphism of their incidence graphs. Given two rank 2 geometries \mathcal{G}_1 and \mathcal{G}_2 , a morphism $\varphi: \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is a *covering* if, for any point p of \mathcal{G}_1 and any line l of \mathcal{G}_1 , φ induces a bijection from the pointset of l to the pointset of $\varphi(l)$ and from the set of lines of \mathcal{G}_1 through p to the set of lines of \mathcal{G}_2 through $\varphi(p)$. If the fibers of a cover have all the same size t then we will say that φ is a *t-fold covering*, \mathcal{G}_1 is called a *t-fold cover* of \mathcal{G}_2 and \mathcal{G}_2 is called a *t-fold quotient* of \mathcal{G}_1 . Note that all coverings of connected geometries are surjective (see [10, 8.3]). In the terminology of graph theory, a covering of geometries is a covering of the incidence graphs of the given geometries. It also induces a covering of the collinearity graphs, satisfying the additional property that, if a, b, c are different points of \mathcal{G}_1 with b and c collinear with a and $\varphi(a), \varphi(b), \varphi(c)$ belonging to a common line of \mathcal{G}_2 , then a, b, c belong to the same line of \mathcal{G}_1 . We warn that, however, if b and c are collinear with a and $\varphi(b)$ and $\varphi(c)$ are mutually collinear but not on the same line as $\varphi(a)$, then b and c need not be collinear in \mathcal{G}_1 .

In Section 2 we shall prove the following:

Theorem 1 *The geometry $\hat{\mathcal{X}}$ is a 2-fold cover of the geometry \mathcal{X} . Explicitly, let $\varphi_0: \hat{\mathcal{X}} \rightarrow \mathcal{X}$ be the mapping defined as follows:*

$$\varphi_0(x) = x^\perp \cap Q_0 \text{ for every point } x \in \hat{P},$$

$$\varphi_0(\hat{l}) = \rho_p \text{ where } p = l^\infty, \text{ for every line } \hat{l} \in \hat{L}.$$

Then φ_0 is a 2-fold covering.

The covering φ_0 defined as above will be called the *canonical* covering from $\widehat{\mathcal{X}}$ to \mathcal{X} . As Q can be recovered from $\widehat{\mathcal{X}}$ (see [13]), every automorphism $\hat{\alpha}$ of $\widehat{\mathcal{X}}$ can be extended to a unique automorphism α of Q stabilizing Q_0 . Let α_0 be the automorphism of Q_0 induced by α . Clearly, $\varphi_0 \circ \hat{\alpha} = \alpha_0 \circ \varphi_0$. It follows that the composition $\varphi := \varphi_0 \circ \hat{\alpha}$ is still a 2-fold covering of \mathcal{X} . Perhaps, all 2-fold coverings from $\widehat{\mathcal{X}}$ to \mathcal{X} arise in this way, but we are not going to investigate this conjecture in this paper.

In Section 3 we determine the cliques of the collinearity graph Γ of \mathcal{X} . A clique of Γ is called *linear* if it is a subset of a line of \mathcal{X} ; it is said to be *non-linear* otherwise. A clique of size i is called an *i-clique*.

As we will prove in Section 3 (Proposition 3.4) a linear clique and a non-linear clique can have at most two vertices in common.

Theorem 2 *All the following hold*

- 1) *Let $q = 2^n$ with n even. Then the maximal cliques of Γ have size q or 4. When $q > 4$ cliques of size q are linear while the maximal cliques of size 4 are non-linear.*
- 2) *Let $q = 2^n > 2$ with n odd. Then the maximal cliques of Γ have size q or 6. The cliques of size q are linear cliques. The maximal cliques of size 6 are non-linear.*
- (3) *In any case, every non-linear 3-clique can be extended to a non-linear 4-clique in $q + 1$ ways.*
- (4) *Let $q = 2^n$ with n odd. Then*
 - (4.i) *Any non-linear 4-clique can be extended to a 5-clique in two ways.*
 - (4.ii) *Any non-linear 5-clique can be extended to a 6-clique in only one way.*

Note that the restrictions $q > 4$ and $q > 2$ in claim 1) and claim 2) of Theorem 2 are essential. Indeed, when $q = 2$ then Γ is a complete graph with 6 vertices. When $q = 4$, all maximal cliques of Γ have size 4.

Denote by $\widehat{\Gamma}$ the collinearity graph of $\widehat{\mathcal{X}}$.

A hexagon $H = (a^1, b^1, c^1, a^2, b^2, c^2, a^1)$ of $\widehat{\mathcal{X}}$ is a 6-circuit of $\widehat{\Gamma}$. Note that H admits a bipartition by 3-sets $\{a^1, b^1, c^1\}$ and $\{a^2, b^2, c^2\}$ as well as a 3-partition by 2-sets $\{a^1, a^2\}$, $\{b^1, b^2\}$ and $\{c^1, c^2\}$ such that vertices in the same pair are on a secant line to Q through n_0 . Vertices in the same pair are called *opposite*. We say that a hexagon is *centric* if all projective lines through opposite vertices are concurrent at the same point called the *center of the hexagon*.

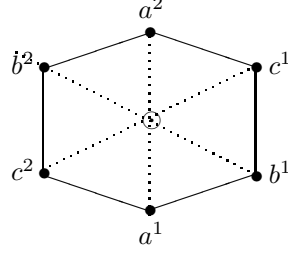


Figure 1

A *cube* C of $\widehat{\mathcal{X}}$ is an induced subgraph of $\widehat{\Gamma}$ with 8 vertices as in the following picture, where adjacencies are represented by thick lines:

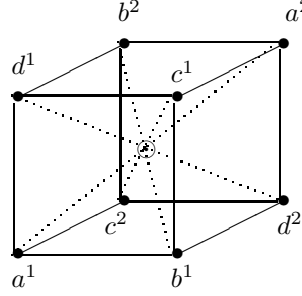


Figure 2

The graph C admits a bipartition in 4-sets $\{a^1, b^1, c^1, d^1\}$, $\{a^2, b^2, c^2, d^2\}$ as well as a 4-partition in pairs $\{a^1, a^2\}$, $\{b^1, b^2\}$, $\{c^1, c^2\}$, $\{d^1, d^2\}$, two vertices in the same pair being called *opposite*. We say that a cube is *centric* if all projective lines through opposite vertices are concurrent at the same point called the *center of the cube*.

Note that a hexagon (respectively, a cube) is the complement of a (3×2) -grid graph (respectively, a (4×2) -grid graph). The bipartition in two classes of size 3 (respectively 4) is one of the two families of lines of the grid and the 3-partition (4-partition) in pairs is the other family. We exploit this observation to define *dodecades*. Consider a (6×2) -grid as follows:

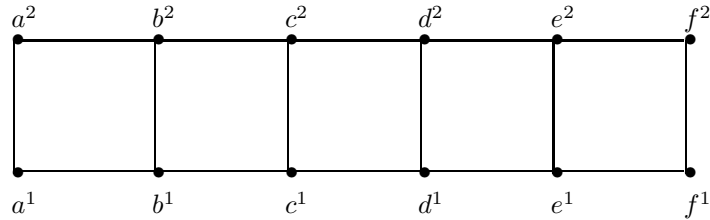


Figure 3

A *dodecade* of $\widehat{\mathcal{X}}$ is an induced subgraph of $\widehat{\Gamma}$ with 12 vertices

$$D = \{a^1, b^1, c^1, d^1, e^1, f^1, a^2, b^2, c^2, d^2, e^2, f^2\}$$

isomorphic to the complement of the collinearity graph of a grid as in Fig. 3. Clearly, D admits a bipartition into two classes of size 6 (corresponding to the two long lines of the grid) as well as a 6-partition into pairs, corresponding to the six short lines. Two points in the same pair are said to be *opposite*. Clearly, any four (respectively, three) pairs of opposite vertices of a dodecade form a cube (a hexagon). We say that a dodecade is *centric* if all the projective lines through opposite vertices are concurrent at the same point called the *center of the dodecade*.

Let $\varphi = \varphi_0 \circ \hat{\alpha}$ where $\varphi_0: \hat{\mathcal{X}} \rightarrow \mathcal{X}$ is the canonical 2-fold covering of \mathcal{X} as in Theorem 1 and $\hat{\alpha} \in \text{Aut}(\hat{\mathcal{X}})$.

For the subgraph Γ_A of Γ induced on a subset A of the vertex set of \mathcal{X} , the subgraph $\hat{\Gamma}_A$ of $\hat{\Gamma}$ induced on the subset $\varphi^{-1}(A)$ of the vertex set of $\hat{\Gamma}$ is called the *preimage* of Γ_A by φ .

Given a path (X_0, X_1, \dots, X_n) in Γ and a vertex $x_0 \in \varphi^{-1}(X_0)$, there exists a unique path (x_0, x_1, \dots, x_n) in $\hat{\Gamma}$ such that $\varphi(x_i) = X_i$ for every $i = 0, 1, \dots, n$. In graph theory and topology the path (x_0, x_1, \dots, x_n) is called the *lifting* of (X_0, X_1, \dots, X_n) at x_0 through φ . By a harmless terminological abuse, we adapt this terminology to the induced subgraph Γ_A of Γ , calling *liftings* of Γ_A the connected components of $\hat{\Gamma}_A$.

The following theorem gives a characterization of the cliques of Γ in terms of their liftings. Recall that n_0 is the nucleus of Q_0 .

Theorem 3 *Let $\varphi = \varphi_0 \circ \hat{\alpha}$ with φ_0 and $\hat{\alpha}$ as above.*

- (a) *The covering φ induces a bijection between the set of all centric hexagons of $\hat{\Gamma}$ with center n_0 and the set of all non-linear 3-cliques in Γ .*
- (b) *The covering φ induces a bijection between the set of all centric cubes of $\hat{\Gamma}$ with center n_0 and the set of all non-linear 4-cliques in Γ .*
- (c) *The covering φ induces a bijection between the set of all centric dodecades of $\hat{\Gamma}$ with center n_0 and the set of all non-linear 6-cliques in Γ .*

Let S be a subset of $\text{PG}(V)$. According to [4], there exists a unique subfield \mathbb{F}_S of \mathbb{F}_q such that S is contained in a projective subgeometry of $\text{PG}(V)$ defined over \mathbb{F}_S and \mathbb{F}_S is minimal with respect to this property. Moreover, the family of projective subgeometries of $\text{PG}(V)$ defined over \mathbb{F}_S and containing S admits a smallest member which we shall denote by $\langle S \rangle_{\mathbb{F}_S}$. We call $\langle S \rangle_{\mathbb{F}_S}$ the \mathbb{F}_S -*span* of S .

If S is a subset of Q , we define the *subgeometry* of Q induced on S to be the pair $Q(S) := (S, \mathcal{L}_S)$ where

$$\mathcal{L}_S := \{l \cap S : l \text{ is a line of } Q \text{ such that } |l \cap S| \geq 2\}.$$

As recalled a few lines above, there exists a unique smallest projective subgeometry $\langle S \rangle_{\mathbb{F}_S}$ of $\text{PG}(V)$ containing S and defined over a subfield \mathbb{F}_S of \mathbb{F}_q . We say that $Q(S)$ is an \mathbb{F}_2 -*subgeometry* of Q if $\mathbb{F}_S = \mathbb{F}_2$.

The next theorem, combined with Theorem 3, gives a characterization of non-linear 3-, 4- and 6-cliques in terms of \mathbb{F}_2 -subgeometries of Q .

Theorem 4 *All the following hold.*

- 1) *A centric hexagon of $\widehat{\mathcal{X}}$ with center n_0 is the subgraph of $\widehat{\Gamma}$ induced on the set-complement $Q(S) \setminus H_0$ of $H_0 = n_0^\perp$ in an \mathbb{F}_2 -subgeometry $Q(S) \cong Q^+(3, 2)$ of Q such that $n_0 \in \langle S \rangle_{\mathbb{F}_2}$.*
- 2) *A centric cube of $\widehat{\mathcal{X}}$ with center n_0 is the subgraph of $\widehat{\Gamma}$ induced on the set-complement $Q(S) \setminus H_0$ of $H_0 = n_0^\perp$ in an \mathbb{F}_2 -subgeometry $Q(S) \cong Q(4, 2)$ of Q such that $n_0 \in \langle S \rangle_{\mathbb{F}_2}$.*
- 3) *A centric dodecade of $\widehat{\mathcal{X}}$ with center n_0 is the subgraph of $\widehat{\Gamma}$ induced on the set-complement $Q(S) \setminus H_0$ of $H_0 = n_0^\perp$ in a \mathbb{F}_2 -subgeometry $Q(S) \cong Q^-(5, 2)$ of Q such that $n_0 \in \langle S \rangle_{\mathbb{F}_2}$.*

The relevance of the parity of n for the existence of maximal non-linear 6-cliques (Theorem 2) is justified by Theorem 3 and Theorem 4. Indeed, every non-linear 6-clique arises from the complement of H_0 in a $(2, 4)$ -subquadrangle $Q^-(5, 2)$ of Q (claim 3 of Theorem 4), but it is well known that $Q^-(5, 2)$ is a subgeometry of $Q^-(5, 2^n)$ if and only if n is odd.

As far as non-linear 4-cliques are concerned, each non-linear 4-clique arises from the complement of a tangent hyperplane in the $(2, 2)$ -subquadrangle $Q(4, 2)$ of $Q^-(5, 2^n)$. Observe that $Q(4, 2)$ is a subgeometry of $Q^-(5, 2^n)$ for any n . So, non-linear 4-cliques exist regardless the parity of n .

2 The semi-partial geometry \mathcal{X}

We will keep the same terminology and notation as in Section 1.

2.1 The quotient geometry $\widehat{\mathcal{X}}/R$

Let R be the equivalence relation defined on the pointset \widehat{P} of the affine generalized quadrangle $\widehat{\mathcal{X}}$ as follows: for $x, y \in \widehat{P}$, put xRy if and only if either $x = y$ or $x^\perp \cap H_0 = y^\perp \cap H_0 (= x^\perp \cap y^\perp)$ (see [10, 8.4]), where \perp is the orthogonality relation defined by Q , as in Section 1. Note that distinct points of $\widehat{\mathcal{X}}$ correspond in R if and only if they have distance 3 in the collinearity graph of $\widehat{\mathcal{X}}$.

For $x \in \widehat{P}$ and $\hat{l} \in \widehat{L}$, let $[x] = \{y \in \widehat{P} : xRy\}$ and $[\hat{l}] := \{[x] : x \in \hat{l}\}$. Given any two distinct lines \hat{l} and \hat{m} of $\widehat{\mathcal{X}}$, we put $\hat{l}R\hat{m}$ if and only if $[\hat{l}] = [\hat{m}]$.

We define the *quotient geometry* $\widehat{\mathcal{X}}/R$ as the point-line geometry having $\{[x] : x \in \widehat{P}\}$ as the pointset and $\{[\hat{l}] : \hat{l} \in \widehat{L}\}$ as the lineset. A point $[x]$ is incident with a line $[\hat{l}]$ if $[x] \in [\hat{l}]$.

We will now give a different equivalent description of the quotient geometry $\widehat{\mathcal{X}}/R$. We recall some well-known facts about the action of the polarity defined by \perp on the subspaces of $\text{PG}(V)$ (see e.g. [7]).

- If x is a point of Q , then x^\perp is a hyperplane tangent to Q at x .

- If S is a 3-dimensional subspace of $\text{PG}(V)$ such that $S \cap Q$ is an elliptic quadric, then S^\perp is a line secant to Q .
- If S is a 3-dimensional subspace of $\text{PG}(V)$ such that $S \cap Q$ is a hyperbolic quadric, then S^\perp is a line disjoint from Q .
- If π_p is a projective plane of $\text{PG}(V)$ such that $\pi_p \cap Q = \{p\}$, then π_p^\perp is a projective plane intersecting Q in two lines meeting at p . Moreover $\pi_p \cap \pi_p^\perp = \{p\}$ and $\langle \pi_p, \pi_p^\perp \rangle = p^\perp$.

Let ν be the unique non-trivial elation of $\text{PG}(V)$ stabilizing Q and having the nucleus n_0 of Q_0 as the center and the hyperplane $H_0 = n_0^\perp$ as the axis. Clearly, ν is an involution.

The relation R can be equivalently defined by means of the collineation ν . More precisely, the orbits of ν on the pointset of $\hat{\mathcal{X}}$ are precisely the equivalence classes of R on \hat{P} . Hence, for any point x of \hat{P} , the equivalence class $[x]$ is precisely the set $\{x, \nu(x)\}$ obtained as the intersection $\langle x, n_0 \rangle \cap Q$, where $\langle x, n_0 \rangle$ is the line (secant to Q) of $\text{PG}(V)$ through x and n_0 . Accordingly, a line $[\hat{l}]$ of $\hat{\mathcal{X}}/R$ corresponds to the set $(l \cup \nu(l)) \setminus (l \cap \nu(l))$ obtained as the intersection $\langle l, \nu(l) \rangle \cap Q$ minus the point $l \cap \nu(l)$, where $l = \hat{l} \cup \{l^\infty\}$ (see (1)). Note also that $l \cap \nu(l) = l^\infty$ and $\langle l, \nu(l) \rangle = \langle l, n_0 \rangle$.

2.2 Proof of Theorem 1

According to the definition of t -fold covering given in Section 1.1 and the facts recalled in Section 2.1, it is easy to see that the projection $\pi: \hat{\mathcal{X}} \rightarrow \hat{\mathcal{X}}/R$ mapping any point x of $\hat{\mathcal{X}}$ to the point $[x]$ of $\hat{\mathcal{X}}/R$ and any line l of $\hat{\mathcal{X}}$ to the line $[l]$ of $\hat{\mathcal{X}}/R$ is a 2-fold covering. Hence

Lemma 2.1 *$\hat{\mathcal{X}}$ is a 2-fold cover of $\hat{\mathcal{X}}/R$.*

Lemma 2.2 *The geometries $\hat{\mathcal{X}}/R$ and \mathcal{X} are isomorphic.*

Proof. Let $\psi_0: \mathcal{X} \rightarrow \hat{\mathcal{X}}/R$ be the map defined on the points and the lines of \mathcal{X} as follows: for $X \in \mathcal{E}$, the perp X^\perp of X in $\text{PG}(5, q)$ is a line of $\text{PG}(5, q)$ through n_0 , secant for Q . We set $\psi_0(X) = X^\perp \cap Q$. For a line ρ_p of \mathcal{X} , let π_p be the unique tangent plane to all $X \in \rho_p$ (see Proposition 1.2). Then π_p^\perp is a plane of $\text{PG}(5, q)$ intersecting Q in two lines l, l' such that $l \cap Q_0 = l' \cap Q_0 = l \cap l' = p$. Then $\psi_0(\rho_p) = [\hat{l}] = [\hat{l}']$. It is easily seen that ψ_0 is an isomorphism of geometries. \square

Theorem 1 follows from Lemma 2.1 and Lemma 2.2, in particular $\varphi_0 := \psi_0^{-1} \circ \pi$ where φ_0 is the canonical covering of \mathcal{X} as defined in Theorem 1.

3 The collinearity graph of \mathcal{X}

In this section we will investigate properties of the collinearity graph Γ of \mathcal{X} . Recall that the vertices of Γ are the elliptic ovoids of Q_0 and two vertices X_1 and X_2 are adjacent if and only if they are tangent, i.e. $|X_1 \cap X_2| = 1$.

Proposition 3.1 *If $q > 2$ then the collinearity graph Γ of \mathcal{X} has diameter 2.*

Proof. The collinearity graph $\widehat{\Gamma}$ of $\widehat{\mathcal{X}}$ has diameter 3 and the classes of R are the pairs of points of $\widehat{\mathcal{X}}$ at mutual distance 3 (see [10, 8.4]). Hence $\widehat{\Gamma}/R$ has diameter at most 2. In fact, it is easy to see that it has diameter exactly 2 except when $q = 2$. By Lemma 2.2 the graph Γ has diameter 2. \square

If $q = 2$, then Γ is a complete graph on 6 vertices. In this case $\widehat{\Gamma}$ is the set complement of a (2×6) -grid and the classes of R are the short lines of the grid. We assume that $q \geq 4$.

In [8] it is proved that Γ is a strongly regular graph with parameters $v = q^2(q^2 - 1)/2$, $k = (q - 1)(q^2 + 1)$, $\lambda = (q - 1)(q + 2)$, and $\mu = 2q(q - 1)$.

Relying on properties of the canonical covering φ_0 we can immediately obtain the parameters (v, k, λ, μ) of Γ . Indeed,

v : The number of vertices of Γ is $|\widehat{P}|/2$.

k : The number of vertices adjacent to a given vertex X of Γ is
 $|\{\text{lines of } \widehat{\mathcal{X}} \text{ through } x\}| \times$
 $\times |\{\text{points different from } x \text{ on any line of } \widehat{\mathcal{X}} \text{ through } x\}|,$
for a point $x \in X^\perp \cap \widehat{P}$, no matter which.

λ : Let X and Y be two adjacent vertices of Γ , i.e. X and Y are two elliptic ovoids of Q_0 tangent at the point $p \in Q_0$. Hence they define a line ρ_p of \mathcal{X} . Then $\varphi_0^{-1}(\rho_p) = \pi_p^\perp \cap Q = \{l^1, l^2\} \setminus \{p\}$ where l^1, l^2 are lines of Q through p . Hence $\varphi_0^{-1}(X) = \{x^1, x^2\} \subset \varphi_0^{-1}(\rho_p)$ and $\varphi_0^{-1}(Y) = \{y^1, y^2\} \subset \varphi_0^{-1}(\rho_p)$. Suppose x^1 and y^1 are on \hat{l}^1 and x^2 and y^2 are on \hat{l}^2 . The number of vertices of Γ adjacent to both X and Y is the same as the number of points $z \in \widehat{P}$ on $\hat{l}^1 \setminus \{x^1, y^1\}$ (equivalently, the number of points on $\hat{l}^2 \setminus \{x^2, y^2\}$). These points bijectively correspond to the ovoids of ρ_p different from X and Y . The number of such ovoids is $q - 2$. We must add the number of points $z \in \widehat{\mathcal{X}}$ at distance 1 from both x^1 and y^2 (equivalently, at distance 1 from both x^2 and y^1). There are q^2 such points. Indeed, on any line of $\widehat{\mathcal{X}}$ through x^1 different from \hat{l}^1 there is a unique point of $\widehat{\mathcal{X}}$ collinear to y^2 . Hence $\lambda = q^2 + q - 2$.

μ : Let X and Y be two non-adjacent vertices of Γ , i.e. X and Y are two elliptic ovoids of Q_0 intersecting in a conic. Suppose $\varphi_0^{-1}(X) = \{x^1, x^2\}$ and $\varphi_0^{-1}(Y) = \{y^1, y^2\}$. The number of vertices of Γ adjacent to both X and Y is the same as the number of points $z \in Q \setminus Q_0$ not collinear with

any point of $X \cap Y$, which are collinear to x^1 and y^1 or to x^2 and y^1 . Let \hat{l} be a line of $\hat{\mathcal{X}}$ through y^1 with $l^\infty \in Y \setminus X$. There are $q^2 - q$ such lines. For each of these $q^2 - q$ lines there are exactly two distinct points adjacent to only one of x^1 or x^2 . Hence $\mu = 2(q^2 - q)$.

Note that there are no elliptic ovoids Z tangent at X and Y at a point in $X \cap Y$ because otherwise X and Y would be on the same line of \mathcal{X} , which is not possible because X and Y are not adjacent in Γ .

In the above calculation of the parameters λ and μ we have also proved two interesting properties of the geometry \mathcal{X} which we will state as propositions:

Proposition 3.2 *Let X and Y be two tangent elliptic ovoids of Q_0 . Take a point $x \in X_1 \setminus X_2$. Then there exists a unique elliptic ovoid of Q_0 through x which is tangent to both X_1 and X_2 .*

Proposition 3.3 *Let X_1 and X_2 be two elliptic ovoids of Q_0 intersecting in a conic. If $x \in X_1 \setminus X_2$ then there exist exactly two distinct elliptic ovoids of Q_0 through x which are both tangent to X_1 and X_2 . If $x \in X_1 \cap X_2$ then there exist no elliptic ovoids of Q_0 through x which are both tangent to X_1 and X_2 .*

3.1 Cliques of Γ and their liftings

In this section we classify cliques in Γ . A clique in Γ consisting of members of \mathcal{X} sharing the same point p is called a *linear clique (based at p)*. A rosette of \mathcal{X} based at p , being a maximal pencil of elliptic ovoids mutually tangent at p , is a maximal linear clique (this is part of claim 1 and 2 of Theorem 2).

Let $\hat{\Gamma}$ denote the collinearity graph of the affine quadrangle $\hat{\mathcal{X}}$, as in Section 1. We recall that $\hat{\Gamma}$ is regular with diameter 3, it contains 4-circuits and every clique of $\hat{\Gamma}$ is contained in a line of $\hat{\mathcal{X}}$. Let $\varphi_0: \hat{\mathcal{X}} \rightarrow \mathcal{X}$ be the canonical covering as in Theorem 1. The fibers of φ_0 are the pairs of points of $\hat{\mathcal{X}}$ at mutual distance 3 (see Section 2.1). We will adopt the following notation. If X is a vertex of Γ , then $\varphi^{-1}(X) = \{x^1, x^2\} (= X^\perp \cap Q)$.

3.2 Preimages of edges and linear cliques

The preimage of an edge in Γ is the union of two disjoint edges in $\hat{\Gamma}$ which are the liftings of that edge. Given a line ρ_p of \mathcal{X} , we have $\varphi_0^{-1}(\rho_p) = \hat{l} \cup \hat{m}$ for two lines \hat{l}, \hat{m} of $\hat{\mathcal{X}}$ such that $l^\infty = m^\infty = p$. The preimage of the subgraph Γ_{ρ_p} induced by Γ on ρ_p is the subgraph induced by $\hat{\Gamma}$ on $\hat{l} \cup \hat{m}$. This graph is the disjoint union of two complete subgraphs, namely \hat{l} and \hat{m} .

Proposition 3.4 *Let \mathcal{C} be an i -clique of Γ , $i \geq 4$, containing a linear 3-clique. Then \mathcal{C} is a linear clique.*

Proof. Let A, B, C, X be four distinct vertices of \mathcal{C} , with A, B, C forming a linear subclique and let ρ be the line of \mathcal{X} on $\{A, B, C\}$.

By way of contradiction suppose that $X \notin \rho$.

Let us lift the ordered closed configuration (X, A, B, C, X) of Γ starting at x^1 recalling that A, B, C are on the same line ρ of \mathcal{X} . Since φ_0 is a covering, the point x^1 is adjacent in $\widehat{\Gamma}$ with exactly one preimage of A , say a^1 . Similarly, a^1 is adjacent with exactly one preimage b^1 of B and b^1 is adjacent with exactly one preimage c^1 of C . Again, c^1 is adjacent with exactly one preimage x^2 of X . Moreover, since φ_0 is a covering, $\varphi_0^{-1}(\rho)$ contains exactly one line \hat{l} of $\widehat{\mathcal{X}}$ incident with all of a^1, b^1, c^1 . On the other hand, $x^1, x^2 \notin \hat{l}$ as $X \notin \rho$.

Since there are no triangles in $\widehat{\mathcal{X}}$, $x^1 \neq x^2$. The path $(x^1, a^1, b^1, c^1, x^2)$ is the lifting of (X, A, B, C, X) at x^1 .

Lift now the closed ordered path (X, A, B, X) of Γ starting at x^1 . Proceeding in the same way as above, we get a path (x^1, a^1, b^1, x^3) where the vertex x^3 must be different from x^1 because there are no triangles in $\widehat{\mathcal{X}}$. Similarly, (x^2, c^1, b^1, x^3) is the lifting of (X, C, B, X) at x^2 . Hence $x^3 \neq x^2$. (see Fig.4).

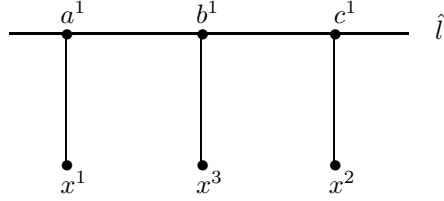


Figure 4

It follows that $\varphi_0^{-1}(X)$ contains at least three distinct points, namely x^1, x^2, x^3 . This is a contradiction because φ_0 is a 2-fold covering. \square

3.3 Liftings of non-linear cliques

We establish the following notation for projective points: $a = [v] = [v_1, \dots, v_n]$ where $v = (v_1, \dots, v_n)$ is a given vector representing the projective point a .

Given two projective points a and b , symbols as $ta + sb$ with t and s scalars are nonsense. However, when particular representatives v and w have been chosen for a and b , we will take the liberty of writing $ta + sb$ for $[tv + sw]$. Similarly, given a quadratic form f , we write $f(a)$ for $f(v)$. Of course, these are notational abuses but they are harmless as far as it is clear which is the representative vector chosen for a given point. In particular, when doing so, we forbid ourselves any rescaling of the given vectors.

In view of Proposition 3.4, in the following sections we only consider non-linear cliques of Γ .

Suppose that the (q, q^2) -generalized quadrangle Q is represented, with respect to a given basis, by the equation $f(x) = 0$ where

$$f((x_1, x_2, x_3, x_4, x_5, x_6)) := x_1x_2 + x_3x_4 + x_5^2 + x_5x_6 + \lambda x_6^2 \quad (2)$$

with λ a given element of \mathbb{F}_q with $Tr(\lambda) = 1$. Recall that the trace function $Tr: \mathbb{F}_q \rightarrow \mathbb{F}_2$, $Tr(x) = \sum_{i=0}^{n-1} x^{2^i}$, ($q = 2^n$), is an \mathbb{F}_2 -linear surjective homo-

morphism and a polynomial $x^2 + xy + \lambda y^2$ is irreducible in \mathbb{F}_q if and only if $Tr(\lambda) = 1$.

We may assume that the nucleus n_0 of Q_0 is represented by the vector $(0, 0, 0, 0, 0, 1)$. Accordingly, $H = n_0^\perp$ is the hyperplane of equation $x_6 = 0$.

A set of four distinct points $\{a, b, c, d\}$ of Q is a *quadrangle* of Q if it is the set of points of a $(1, 1)$ -subquadrangle of Q . Hence

$$q(a) = q(b) = q(c) = q(d) = 0; \quad a \perp b \perp c \perp d \perp a; \quad a \not\perp c; \quad b \not\perp d. \quad (3)$$

We first need a rather technical lemma.

Lemma 3.5 *Let λ, μ be two elements of \mathbb{F}_q . Suppose $Tr(\lambda) = 1$ and $\mu \neq 1$. Then the equation $x^2 + xy + \lambda y^2 + \mu = 1$, in the unknowns x, y , admits $q + 1$ distinct solutions in \mathbb{F}_q^2 .*

Proof. If $y = 0$ then the pair $(x, y) = (\mu^{2^{n-1}} + 1, 0)$ is the unique solution of $x^2 + xy + \lambda y^2 + \mu = 1$. If $y \neq 0$ then the equation $x^2 + xy + \lambda y^2 + \mu = 1$ is soluble if and only if the equation $t^2 + t + \lambda + (\mu + 1)/y^2 = 0$ in the unknown $t := x/y$ is soluble. This latter requirement is equivalent to $Tr(\lambda + (\mu + 1)/y^2) = 0$. Since

$$Tr(\lambda + (\mu + 1)/y^2) = Tr(\lambda) + Tr((\mu + 1)/y^2) = 1 + Tr((\mu + 1)/y^2),$$

the equation $t^2 + t + (\lambda + (\mu + 1)/y^2) = 0$ admits solutions if and only if $Tr((\mu + 1)/y^2) = 1$.

As, the trace function Tr is a surjective homomorphism of \mathbb{F}_2 -vector spaces there are $q/2$ distinct values of y such that $Tr((\mu + 1)/y^2) = 1$ (see e.g. [9]). For each of these $q/2$ values of y , there are two distinct solutions of $t^2 + t + \lambda + (\mu + 1)/y^2 = 0$. So, if $y \neq 0$, there are $2 \cdot \frac{q}{2}$ distinct pairs (x, y) which satisfy $x^2 + xy + \lambda y^2 + \mu = 1$. \square

3.3.1 Non linear 3-cliques and 4-cliques

Hexagons, cubes and dodecades of $\widehat{\mathcal{X}}$ and their centers, if any, have been defined in Section 1.1. Clearly, those definitions can be immediately generalized to Q , by replacing $\widehat{\Gamma}$ with the collinearity graph of Q . In particular, a hexagon (cube, dodecade) of Q is *centric* if the projective lines joining pairs of opposite points are concurrent in one point, called the *center* of the hexagon (cube, dodecade). These more general notions will be used from times to times in the sequel.

Theorem 3.6 *The preimage in $\widehat{\Gamma}$ of a non-linear 3-clique of Γ is a centric hexagon of \mathcal{X} with the nucleus n_0 of Q_0 as the center.*

Proof. Let $\mathcal{C} = \{A, B, C\}$ be a non-linear 3-clique of Γ . Let us lift the ordered closed configuration (A, B, C, A) of \mathcal{C} starting at $a^1 \in \varphi_0^{-1}(A)$. We get a path (a^1, b^1, c^1, a^2) of $\widehat{\Gamma}$, where $a^1 \neq a^2$ since in $\widehat{\mathcal{X}}$ there are no proper triangles. Consider now the lifting of (A, B, C, A) at a^2 . We get a path (a^2, b^2, c^2, a^3) where $b^2 \neq b^1$ and $c^2 \neq c^1$ because there are no proper triangles in $\widehat{\mathcal{X}}$ and a^3 is a preimage of A , different from a^2 . As $|\varphi_0^{-1}(A)| = 2$, necessarily $a^3 \neq a^1$. So, by

pasting $\{a^1, b^1, c^1, a^2\}$ with $\{a^2, b^2, c^2, a^1\}$, we obtain a proper hexagon H . As φ_0 is a 2-fold covering, $H = \varphi_0^{-1}(\mathcal{C})$, namely \mathcal{C} lifts to H .

By definition of the canonical covering φ_0 , each of the pairs $\{a^1, a^2\}$, $\{b^1, b^2\}$, $\{c^1, c^2\}$ is on a secant line to Q through n_0 . \square

Clearly, every centric hexagon of $\widehat{\Gamma}$ with center n_0 is mapped by φ_0 onto a non-linear 3-clique of Γ . This observation combined with Theorem 3.6 proves claim a) of Theorem 3.

Theorem 3.7 *The preimage of a non-linear 4-clique of Γ is a centric cube of $\widehat{\mathcal{X}}$ with the nucleus n_0 of Q_0 as the center.*

Proof. Let $\mathcal{C} = \{A, B, C, D\}$ be a non-linear 4-clique of Γ . Let $\varphi_0^{-1}(A) = \{a^1, a^2\}$, $\varphi_0^{-1}(B) = \{b^1, b^2\}$, $\varphi_0^{-1}(C) = \{c^1, c^2\}$, $\varphi_0^{-1}(D) = \{d^1, d^2\}$.

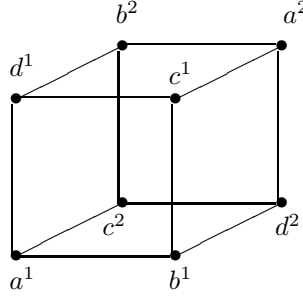
Let us lift the ordered closed path (A, B, C, D, A) of \mathcal{C} starting at a^1 . We get a path $\mathcal{C}^1 = (a^1, b^1, c^1, d^1, a^i)$ where $i = 1$ or $i = 2$. Suppose $i = 2$, namely $\mathcal{C}^1 = (a^1, b^1, c^1, d^1, a^2)$. Then consider the lifting of the ordered closed path (A, B, C, A) starting at a^1 . We get either (a^1, b^1, c^1, a^1) or (a^1, b^1, c^1, a^2) . In the first case we obtain a proper triangle in $\widehat{\mathcal{X}}$, which cannot be. In the second case (a^1, b^1, c^1, a^2) is a triangle in $\widehat{\mathcal{X}}$, again a contradiction.

Hence $i = 1$, namely $\mathcal{C}^1 = (a^1, b^1, c^1, d^1, a^1)$.

Keeping in mind that the liftings to $\widehat{\Gamma}$ of a vertex-subset of Γ cannot contain any proper triangle of $\widehat{\mathcal{X}}$, we have the following:

- a) the lifting of (A, B, C, A) starting at a^1 is (a^1, b^1, c^1, a^2) ;
- b) the lifting of (A, C, D, A) starting at a^1 is (a^1, c^2, d^2, a^2) ;
- c) the lifting of (B, C, A, B) starting at b^1 is (b^1, c^1, a^2, b^2) ;
- d) the lifting of (B, D, A, B) starting at b^1 is (b^1, d^2, a^2, b^2) ;
- e) the lifting of (D, B, C, D) starting at d^1 is (d^1, b^2, c^2, d^2) .

We finally get the cube as below (compare Fig. 2).



By definition of the covering φ_0 , each of the pairs $\{a^1, a^2\}$, $\{b^1, b^2\}$, $\{c^1, c^2\}$, $\{d^1, d^2\}$ is on a secant line to Q through n_0 . \square

Clearly, every centric cube of $\widehat{\Gamma}$ with center n_0 is mapped by φ_0 onto a non-linear 4-clique of Γ . This observation combined with Theorem 3.6 proves claim b) of Theorem 3.

Let \mathcal{F} be a quadrangle of Q . We denote by $I(\mathcal{F})$ the set of points $p \in \text{PG}(V) \setminus Q$ such that $p^\perp \cap \mathcal{F} = \emptyset$ and there exists a cube in the affine quadrangle $Q \setminus p^\perp$ with center p having \mathcal{F} as one of its faces. Clearly, such a cube is uniquely determined by \mathcal{F} and p .

Lemma 3.8 *Let $\bar{\mathcal{F}} = \{\bar{a}^1, \bar{b}^1, \bar{c}^1, \bar{d}^1\}$ be the quadrangle of Q with*

$$\bar{a}^1 = [1, 0, 0, 0, 0, 0], \quad \bar{b}^1 = [0, 0, 1, 0, 0, 0], \quad \bar{c}^1 = [0, 1, 0, 0, 0, 0], \quad \bar{d}^1 = [0, 0, 0, 1, 0, 0].$$

Then

$$I(\bar{\mathcal{F}}) = \left\{ \left[u, \frac{1}{u}, v, \frac{1}{v}, r, s \right] : r^2 + rs + \lambda s^2 = 1, uv \neq 0, u, v \in \mathbb{F}_q \right\} \quad (4)$$

and

$$|I(\bar{\mathcal{F}})| = (q-1)^2(q+1) = (q-1)(q^2-1). \quad (5)$$

Proof. Let $\bar{p} = [p_1, p_2, p_3, p_4, r, s] \in I(\bar{\mathcal{F}})$. Since \bar{p} does not belong to Q , we have $f(\bar{p}) \neq 0$ (with $f(x)$ as in Equation (2)). Without loss of generality we can assume $f(\bar{p}) = 1$, i.e.

$$p_1 p_2 + p_3 p_4 + r^2 + rs + \lambda s^2 = 1. \quad (6)$$

Since $\bar{p}^\perp \cap \bar{\mathcal{F}} = \emptyset$, \bar{p} is not collinear in Q with any element of $\bar{\mathcal{F}}$. This is equivalent to $p_2 \neq 0, p_4 \neq 0, p_1 \neq 0, p_3 \neq 0$.

Suppose that $\bar{a}^2, \bar{b}^2, \bar{c}^2, \bar{d}^2$ are points of Q so that $\bar{C} = \{\bar{a}^i, \bar{b}^i, \bar{c}^i, \bar{d}^i\}_{i=1,2}$ is a centric cube of $Q \setminus \bar{p}^\perp$ with center \bar{p} . Recall that the exponent 1 and 2 of the same symbol refer to opposite points, that the collinearities between points in \bar{C} are as described in Fig. 2 and each of $\bar{a}^2, \bar{b}^2, \bar{c}^2, \bar{d}^2$ are on a line joining \bar{p} to $\bar{a}^1, \bar{b}^1, \bar{c}^1, \bar{d}^1$ respectively. Then, in terms of coordinates, we have

$$\bar{a}^2 = \bar{a}^1 + t_a \bar{p}, \quad \bar{b}^2 = \bar{b}^1 + t_b \bar{p}, \quad \bar{c}^2 = \bar{c}^1 + t_c \bar{p}, \quad \bar{d}^2 = \bar{d}^1 + t_d \bar{p}$$

for suitable $t_a, t_b, t_c, t_d \in \mathbb{F}_q \setminus \{0\}$. By hypothesis, plugging in the coordinates of $\bar{a}^1, \bar{b}^1, \bar{c}^1, \bar{d}^1$, we have

$$\bar{a}^2 = [1 + t_a p_1, t_a p_2, t_a p_3, t_a p_4, t_a r, t_a s]; \quad \bar{b}^2 = [t_b p_1, t_b p_2, 1 + t_b p_3, t_b p_4, t_b r, t_b s];$$

$$\bar{c}^2 = [t_c p_1, t_c p_2 + 1, t_c p_3, t_c p_4, t_c r, t_c s]; \quad \bar{d}^2 = [t_d p_1, t_d p_2, t_d p_3, 1 + t_d p_4, t_d r, t_d s].$$

Translating in terms of coordinates the collinearities in Q for \bar{C} to be a cube with center \bar{p} , we have

$$\left. \begin{aligned} \bar{a}^1 \perp \bar{c}^2 : t_c p_2 + 1 &= 0; & \bar{b}^1 \perp \bar{d}^2 : t_d p_4 + 1 &= 0; \\ \bar{c}^1 \perp \bar{a}^2 : t_a p_1 + 1 &= 0; & \bar{d}^1 \perp \bar{b}^2 : t_b p_3 + 1 &= 0; \\ \bar{a}^2 \perp \bar{b}^2 : t_a t_b p_2 p_1 + t_a t_b p_3 p_4 &= 0; & \bar{b}^2 \perp \bar{c}^2 : t_b t_c p_1 p_2 + t_b t_c p_3 p_4 &= 0; \\ \bar{c}^2 \perp \bar{d}^2 : t_c t_d p_1 p_2 + t_c t_d p_3 p_4 &= 0; & \bar{d}^2 \perp \bar{a}^2 : t_a t_d p_1 p_2 + t_a t_d p_3 p_4 &= 0. \end{aligned} \right\} \quad (7)$$

The last four equations in (7) give $p_1p_2 + p_3p_4 = 0$. Hence, Equation (6) becomes

$$r^2 + rs + \lambda s^2 = 1. \quad (8)$$

Since $\bar{a}^2, \bar{b}^2, \bar{c}^2, \bar{d}^2$ are points of Q , we have $q(\bar{a}^2) = q(\bar{b}^2) = q(\bar{c}^2) = q(\bar{d}^2) = 0$, i.e. $p_2 = t_a, p_4 = t_b, p_1 = t_c$ and $p_3 = t_d$. The first four equations in (7) give $p_1p_2 = 1 = p_3p_4$.

Put $u := p_1$ and $v := p_3$. Then $\bar{p} = [u, 1/u, v, 1/v, r, s]$ where $uv \neq 0$ and r, s satisfy Equation (8).

By Lemma 3.5 with $\mu = 0$, Equation (8) admits $q+1$ distinct solutions. The lemma is proved. \square

We will refer to the quadrangle $\bar{\mathcal{F}}$ as in Lemma 3.8 as the *fundamental quadrangle of Q* . By the proof of Lemma 3.8, the coordinates of the points of the cube $C_{\bar{p}}$ having $\bar{\mathcal{F}}$ as a face and $\bar{p} \in I(\bar{\mathcal{F}})$ as the center are the following

$$\left. \begin{array}{ll} \bar{a}^1 = [1, 0, 0, 0, 0, 0], & \bar{b}^1 = [0, 0, 1, 0, 0, 0], \\ \bar{c}^1 = [0, 1, 0, 0, 0, 0], & \bar{d}^1 = [0, 0, 0, 1, 0, 0], \\ \bar{a}^2 = [0, \frac{1}{u^2}, \frac{v}{u}, \frac{1}{uv}, \frac{r}{u}, \frac{s}{u}], & \bar{b}^2 = [\frac{u}{v}, \frac{1}{uv}, 0, \frac{1}{v^2}, \frac{r}{v}, \frac{s}{v}], \\ \bar{c}^2 = [u^2, 0, uv, \frac{u}{v}, ur, us] & \bar{d}^2 = [uv, \frac{v}{u}, v^2, 0, vr, vs] \\ \bar{p} = [u, 1/u, v, 1/v, r, s]. \end{array} \right\} \quad (9)$$

where $uv \neq 0$ and r, s satisfy Equation (8).

Corollary 3.9 *Every quadrangle of Q can be extended to $(q-1)^2(q+1)$ centric cubes of Q .*

Proof. Straightforward, by (5) of Lemma 3.8 and the transitivity of $SO^-(6, q)$ on the set of quadrangles of Q . \square

Corollary 3.10 Γ *admits non-linear 4-cliques.*

Proof. By Theorem 3.7, if a non-linear 4-clique exists then it can be uniquely lifted to a centric cube with center n_0 . Hence existence of non-linear 4-cliques is equivalent to the existence of centric cubes with center n_0 . By Lemma 3.8, there exists centric cubes having $\bar{\mathcal{F}}$ as a face. Let $C_{\bar{p}}$ one of these cubes and let \bar{p} its center.

The group $SO^-(6, q) \leq \text{Aut}(Q)$ is transitive on the set of points of $PG(5, q) \setminus Q$. Then there exists $g \in SO^-(6, q)$ such that $g(\bar{p}) = n_0$. The cube $g(C_{\bar{p}})$ has the required properties. \square

The following theorem is claim 3) of Theorem 2.

Theorem 3.11 *Every non-linear 3-clique of Γ can be extended to $q+1$ non-linear 4-cliques.*

Proof. By Theorem 3.6 and 3.7, each non-linear 3-clique can be extended to $q + 1$ non-linear 4-cliques if and only if each centric hexagon of $\widehat{\mathcal{X}}$ can be extended to $q + 1$ centric cubes of $\widehat{\mathcal{X}}$ having the same center as the hexagon.

Let $H = \{a^i, b^i, c^i\}_{i=1,2}$ be a centric hexagon of Q , with vertices marked as in Fig. 1.

Since the group $SO^-(6, q)$ is transitive on the set of paths of the collinearity graph of Q of length 2 we can suppose that a^1, b^1, c^1 are indeed the vertices $\bar{a}^1, \bar{b}^1, \bar{c}^1$ as in the fundamental quadrangle $\widehat{\mathcal{F}}$, i.e. a^1, b^1, c^1 are represented by the vectors $(1, 0, 0, 0, 0, 0)$, $(0, 0, 1, 0, 0, 0)$, and $(0, 1, 0, 0, 0, 0)$ respectively.

Let us determine the opposite points a^2, b^2, c^2 . In order to do that, let $p = [p_1, p_2, p_3, p_4, p_5, p_6]$ be the center of H . Hence, p is not collinear in Q with any of a^1, b^1, c^1 . In terms of coordinates, this fact is equivalent to $p_1, p_2, p_3 \neq 0$. Since p is a point of $\text{PG}(V)$ not in Q , $f((p_1, p_2, p_3, p_4, p_5, p_6)) \neq 0$. Without loss of generalities we can suppose $f((p_1, p_2, p_3, p_4, p_5, p_6)) = 1$, i.e. $p_1 p_2 + p_3 p_4 + p_5^2 + p_5 p_6 + \lambda p_6^2 = 1$. Then

$$\begin{aligned} a^2 &= [\mu_a(1, 0, 0, 0, 0, 0) + (p_1, p_2, p_3, p_4, p_5, p_6)] = [\mu_a + p_1, p_2, p_3, p_4, p_5, p_6]; \\ b^2 &= [\mu_b(0, 0, 1, 0, 0, 0) + (p_1, p_2, p_3, p_4, p_5, p_6)] = [p_1, p_2, \mu_b + p_3, p_4, p_5, p_6]; \\ c^2 &= [\mu_c(0, 1, 0, 0, 0, 0) + (p_1, p_2, p_3, p_4, p_5, p_6)] = [p_1, \mu_c + p_2, p_3, p_4, p_5, p_6]. \end{aligned}$$

Since a^2, b^2, c^2 are points of Q and recalling that $f(p) = 1$ by assumption, we get $\mu_a = 1/p_2$; $\mu_b = 1/p_4$; $\mu_c = 1/p_1$, so

$$\left. \begin{aligned} a^2 &= [1/p_2 + p_1, p_2, p_3, p_4, p_5, p_6]; \\ b^2 &= [p_1, p_2, 1/p_4 + p_3, p_4, p_5, p_6]; \\ c^2 &= [p_1, 1/p_1 + p_2, p_3, p_4, p_5, p_6]. \end{aligned} \right\} \quad (10)$$

Translating in terms of coordinates the collinearity relations in H (see Fig. 1), we have

$$\frac{1}{p_2} + p_1 = 0 \text{ because } a^2 \perp c^1 \text{ and } \frac{1}{p_1} + p_2 = 0 \text{ because } c^2 \perp a^1.$$

Hence $p_2 = 1/p_1$. Note that the relations $b^2 \perp a^2$ and $b^1 \perp c^1$ are automatically satisfied.

Hence we get for Equations (10):

$$\left. \begin{aligned} a^2 &= [0, 1/p_1, p_3, p_4, p_5, p_6]; \\ b^2 &= [p_1, 1/p_1, 1/p_4 + p_3, p_4, p_5, p_6]; \\ c^2 &= [p_1, 0, p_3, p_4, p_5, p_6]. \end{aligned} \right\} \quad (11)$$

and $p = [p_1, 1/p_1, p_3, p_4, p_5, p_6]$ with $p_3 p_4 + p_5^2 + p_5 p_6 + \lambda p_6^2 = 0$.

Let $d^1 = [d_1, d_2, d_3, d_4, d_5, d_6]$ and $d^2 = \mu d^1 + p$ ($\mu \in \mathbb{F}_q$) be two points of Q . If we require the set $H \cup \{d^1, d^2\}$ be a cube with center p we obtain the following conditions on the coordinates of d^1 and d^2 : $d_2 = 0$ because $d^1 \perp a^1$ and $d_2 = 0$ because $d^1 \perp c^1$. Also $d_4 \neq 0$ since $d^1 \not\perp b^1$. Hence we can assume $d_4 = 1$. Therefore $d^1 = [0, 0, d_3, 1, d_5, d_6]$. Since $d^1 \perp b^2$ we have

$$d_3 p_4 = \frac{1}{p_4} + p_3 + d_5 p_6 + d_6 p_5. \quad (12)$$

Observe that $d^1 \not\perp b^1$. Indeed

$$\alpha((p_1, 1/p_1, p_3, p_4, p_5, p_6), (0, 0, d_3, 1, d_5, d_6)) = 1/p_4 \neq 0,$$

where α is the bilinear form associated to f . Since d^2 is a point of Q , we get

$$d_3 = d_5^2 + d_5 d_6 + \lambda d_6^2. \quad (13)$$

Let us turn to the opposite point $d^2 = \mu d^1 + p$ of d^1 . The fact that d^2 is a point of Q is equivalent to $1 + \mu/p_4 = 0$, hence $\mu = p_4$. So,

$$d^2 = [p_1, 1/p_1, p_4 d_3 + p_3, 0, p_4 d_5 + p_5, p_4 d_6 + p_6].$$

By (12), we get

$$d^2 = [p_1, 1/p_1, p_5 d_6 + d_5 p_6 + 1/p_4, 0, p_4 d_5 + p_5, p_4 d_6 + p_6].$$

Note that the relations $d^2 \perp b^1$, $d^2 \perp a^2$, $d^2 \perp c^2$ are automatically satisfied.

By conditions (12) and (13) we now have

$$\left(\frac{1}{p_4}\right)^2 + \frac{p_3}{p_4} + d_5 \frac{p_6}{p_4} + d_6 \frac{p_5}{p_4} + d_5^2 + d_5 d_6 + \lambda d_6^2 = 0$$

which is equivalent to

$$\begin{aligned} & \left(d_5 + \frac{p_5}{p_4}\right)^2 + \left(d_5 + \frac{p_5}{p_4}\right) \left(d_6 + \frac{p_6}{p_4}\right) + \lambda \left(d_6 + \frac{p_6}{p_4}\right)^2 + \\ & + \left(\frac{p_5}{p_4}\right)^2 + \frac{p_5 p_6}{p_4^2} + \lambda \left(\frac{p_6}{p_4}\right)^2 + \left(\frac{1}{p_4}\right)^2 + \left(\frac{p_3}{p_4}\right) = 0. \end{aligned}$$

Since

$$\begin{aligned} & \left(\frac{p_5}{p_4}\right)^2 + \frac{p_5 p_6}{p_4^2} + \lambda \left(\frac{p_6}{p_4}\right)^2 + \left(\frac{1}{p_4}\right)^2 + \left(\frac{p_3}{p_4}\right) = \\ & \frac{1 + p_3 p_4 + p_5^2 + p_5 p_6 + \lambda p_6^2}{p_4^2} = \frac{f((p_1, p_2, p_3, p_4, p_5, p_6))}{p_4^2} = \frac{1}{p_4^2} \neq 0 \end{aligned}$$

we can apply Lemma 3.5 with $x := d_5 + \frac{p_5}{p_4}$, $y := d_6 + \frac{p_6}{p_4}$, $\mu = \frac{1}{p_4^2} + 1$ to obtain $q + 1$ different choices for d^1 . The theorem is proved. \square

Proposition 3.12 *The following hold.*

- a) *The number of distinct non-linear 3-cliques of Γ is $N_3 = \frac{(q^4-1)(q-1)q^4}{12}$.*
- b) *The number of distinct non-linear 4-cliques of Γ is $N_4 = \frac{(q^4-1)(q^2-1)q^4}{48}$.*

Proof. We first prove Claim b). By Theorem 3.7, the lifting of a non-linear 4-clique is a centric cube C . Let us double count the set of pairs $\{(p, C)\}$ where p is a point not in Q and C is a cube of $Q \setminus p^\perp$ with center p . Hence $N_4 \cdot (\# \text{ of points not in } Q) = N_4 \cdot \frac{q^6 - 1 - (q^3 + 1)(q^2 - 1)}{q - 1} = N := (\# \text{ of centric cubes in } Q)$.

The total number N of centric cubes in Q can be computed as follows:

$$(\# \text{ of cubes with a given quadrangle as a face}) \cdot (\# \text{ of quadrangles}) = N \cdot (\# \text{ of quadrangles in a given cube}).$$

Counting the number of choices for the vertices of a quadrangle of Q , it is not difficult to see that there are $(q^3 + 1)(q^2 + 1)(q + 1)q^6/8$ quadrangles in Q . By Corollary 3.8, there are $(q - 1)^2(q + 1)$ centric cubes through a given quadrangle. Since every cube has 6 faces, we obtain $N = (q^3 + 1)(q^4 - 1)(q^2 - 1)q^6/48$.

It follows that the number of cubes of Q with a given center is

$$N_4 = N \cdot \frac{q - 1}{(q^6 - 1) - (q^3 + 1)(q^2 - 1)} = \frac{(q^4 - 1)(q^2 - 1)q^4}{48}.$$

We now turn to claim a). By Theorem 3.6, the lifting of a non-linear 3-clique is a centric hexagon. Moreover, every non-linear 3-clique is contained in $q + 1$ non-linear 4-cliques, by Theorem 3.11. Hence

$$N_3 \cdot (\# \text{ of non-linear 4-cliques on a non-linear 3-clique}) = N_4 \cdot (\# \text{ of non-linear 3-cliques in a non-linear 4-clique}).$$

By Part b) of this Proposition, $N_4 = \frac{(q^4 - 1)(q^2 - 1)q^4}{48}$. Since there are four non-linear 3-cliques contained in a 4-clique, the proposition is proved. \square

Note that claim a) of Proposition 3.12 also easily follows by the parameters v, k, λ, μ of Γ as recalled at the beginning of Section 3.

3.3.2 Non linear 5-cliques and 6-cliques

We have defined dodecades of $\widehat{\mathcal{X}}$ in Section 1.1. Decades can be defined in a similar way. Explicitly, a *decade* of $\widehat{\mathcal{X}}$ is a subgraph of $\widehat{\Gamma}$ with 10 vertices

$$T = \{a^1, b^1, c^1, d^1, e^1, a^2, b^2, c^2, d^2, e^2\}$$

isomorphic to the complement of the collinearity graph of a (5×2) -grid as follows:

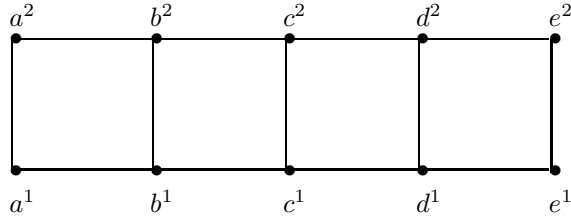


Figure 5

The graph T admits a bipartition in two classes of size 5 (corresponding to the two long lines of the grid) as well as a 5-partition in pairs, corresponding to the five short lines. Two points in the same pair are said to be *opposite*. Clearly, any four (three) pairs of opposite vertices of a decade form a cube (a hexagon). We say that a decade is *centric* if all the projective lines through opposite vertices are concurrent at the same point called the *center of the decade*.

Proposition 3.13 *The following hold.*

- i) *Every non-linear 5-clique of Γ lifts through φ_0 to a centric decade in $\widehat{\Gamma}$ with the nucleus n_0 of Q_0 as the center.*
- ii) *Every non-linear 6-clique of Γ lifts through φ_0 to a centric dodecade of $\widehat{\Gamma}$ with the nucleus n_0 of Q_0 as the center.*

Proof. We will only prove claim ii). The proof of claim i) is similar.

Let $\mathcal{C} = \{a, b, c, d, e, f\}$ be a non-linear 6-clique of Γ . Consider the 4-cliques $\{a, b, c, d\}$ and $\{a, f, e, c\}$ of \mathcal{C} and denote their liftings by $\{a^i, b^i, c^i, d^i\}_{i=1,2}$ and $\{a^i, f^i, e^i, c^i\}_{i=1,2}$, respectively. By Theorem 3.7, the lifting of any non-linear 4-subclique of \mathcal{C} is a centric cube of $\widehat{\mathcal{X}}$ with center n_0 . Hence we can lift \mathcal{C} to a subgraph of $\widehat{\Gamma}$ which is the complement of the following graph

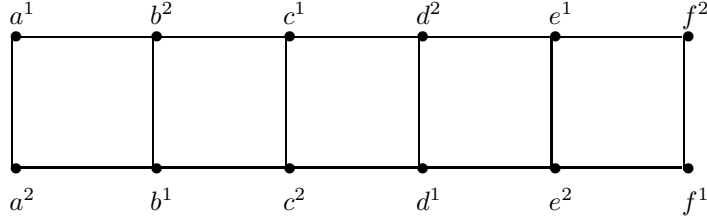


Fig. 6

Clearly, the complement of the graph in Fig. 6 is a dodecade. By definition of the canonical covering φ_0 , each of the opposite pairs $\{a^1, a^2\}$, $\{b^1, b^2\}$, $\{c^1, c^2\}$, $\{d^1, d^2\}$, $\{e^1, e^2\}$, $\{f^1, f^2\}$ is on a secant line to Q through n_0 . \square

Every centric dodecade of $\widehat{\Gamma}$ with center n_0 is mapped by φ_0 onto a non-linear 6-clique of Γ . This observation combined with claim ii) of Proposition 3.13 proves claim c) of Theorem 3.

The following theorem comprises claims 1) and 2) of Theorem 2.

Theorem 3.14 *The following hold.*

- a) *The graph Γ admits non-linear 5-cliques if and only if $q = 2^n$, n odd.*
- b) *Let $q = 2^n$, n odd. Then every 5-clique is contained in a 6-clique.*
- c) *There exist no k -cliques with $k > 6$.*

Proof. Let $C := \{a^i, b^i, c^i, d^i\}_{i=1,2}$ be the lifting of a non-linear 4-clique of Γ . By Theorem 3.7, C is a centric cube of $\widehat{\mathcal{X}}$. By the transitivity of the action of $SO^-(6, q)$ on quadrangles of Q and by the proof of Lemma 3.8 we can suppose without loss of generalities to choose the cube $C_{\bar{p}}$ as in (9) of Section 3.3.1. In particular, $C_{\bar{p}}$ has center $\bar{p} = [u, 1/u, v, 1/v, r, s]$ with $uv \neq 0$ and $r^2 + rs + \lambda s^2 = 1$.

We want to enlarge $C_{\bar{p}}$ in such a way to get a lifting of a k -clique of Γ with $k \geq 5$.

Let $\bar{e}^1 = [e_{1,i}]_{i=1}^6$ and $\bar{e}^2 = [e_{2,i}]_{i=1}^6$ be two points of Q . The conditions on the coordinates of \bar{e}^i ($i = 1, 2$) for the set $C_{\bar{p}} \cup \{\bar{e}^i\}_{i=1}^2$ to be a decade of $Q \setminus \bar{p}^\perp$ with center \bar{p} are the following.

$$\left. \begin{aligned} \bar{b}^1 \perp \bar{e}^1 : \quad e_{1,4} &= 0; & \bar{d}^1 \perp \bar{e}^1 : \quad e_{1,3} &= 0; \\ \bar{a}^2 \perp \bar{e}^1 : \quad e_{1,1} &= e_{1,5}us + e_{1,6}ur; & \bar{c}^2 \perp \bar{e}^1 : \quad e_{1,2} &= e_{1,5}s/u + e_{1,6}r/u; \\ \bar{a}^1 \perp \bar{e}^2 : \quad e_{2,2} &= 0; & \bar{c}^1 \perp \bar{e}^2 : \quad e_{2,1} &= 0; \\ \bar{b}^2 \perp \bar{e}^2 : \quad e_{2,3} &= e_{2,5}sv + e_{2,6}rv; & \bar{d}^2 \perp \bar{e}^2 : \quad e_{2,4} &= e_{2,5}s/v + e_{2,6}r/v. \end{aligned} \right\} \quad (14)$$

Since \bar{e}^1 and \bar{e}^2 are points of Q , we have $f(\bar{e}^1) = 0$ and $f(\bar{e}^2) = 0$. Hence

$$e_{1,5}^2(s^2 + 1) + e_{1,5}e_{1,6} + e_{1,6}^2(r^2 + \lambda) = 0, \quad (15)$$

$$e_{2,5}^2(s^2 + 1) + e_{2,5}e_{2,6} + e_{2,6}^2(r^2 + \lambda) = 0, \quad (16)$$

The points \bar{e}^1, \bar{e}^2 and \bar{p} must be on the same line. Hence

$$e_{2,5} = e_{1,5}(1 + rs) + e_{1,6}r^2 \quad \text{and} \quad e_{2,6} = e_{1,5}s^2 + e_{1,6}(1 + rs). \quad (17)$$

By plugging $e_{2,5}$ and $e_{2,6}$ of Equation (17) in Equation (16) we have

$$e_{1,5}^2((r^2 + rs + \lambda s^2)s^2 + 1) + e_{1,5}e_{1,6} + e_{1,6}^2(r^2 + rs + \lambda s^2) + \lambda = 0.$$

Since $r^2 + rs + \lambda s^2 = 1$, the above equation gives us back Equation (15). So, Equation (16) is already implicit in Equation (15) hence we can disregard it.

The coordinates $e_{1,i}$ of \bar{e}^1 for $1 \leq i \leq 4$ and the coordinates $e_{2,i}$ of \bar{e}^2 for $1 \leq i \leq 4$ depend on $e_{1,5}$ and $e_{1,6}$. Hence we write $\bar{e}_{(e_{1,5}, e_{1,6})}^1$ for \bar{e}^1 and $\bar{e}_{(e_{1,5}, e_{1,6})}^2$ for \bar{e}^2 so that to emphasize the dependance of \bar{e}^1 and \bar{e}^2 on $e_{1,5}$ and $e_{1,6}$.

Explicitly, we have

$$\bar{e}_{(e_{1,5}, e_{1,6})}^1 = (e_{1,5}us + e_{1,6}ur, e_{1,5}\frac{s}{u} + e_{1,6}\frac{r}{u}, 0, 0, e_{1,5}, e_{1,6}) \quad (18)$$

and

$$\bar{e}_{(e_{1,5}, e_{1,6})}^2 = (0, 0, e_{1,5}sv + e_{1,6}rv, \frac{se_{1,5}}{v} + \frac{re_{1,6}}{v}, e_{1,5}(1 + rs) + e_{1,6}r^2, e_{1,5}s^2 + e_{1,6}(1 + rs)). \quad (19)$$

Note that the possibility to enlarge $C_{\bar{p}}$ in order to get a lifting of a non-linear k -clique with $k \geq 5$ is equivalent to the existence of at least two points \bar{e}^1 and

\bar{e}^2 as above. The number of choices for the pair (\bar{e}^1, \bar{e}^2) determines the size of the k -clique. After having implemented all the collinearity relations among the points $C_{\bar{p}} \cup \{\bar{e}^1, \bar{e}^2\}$ to get a centric decade, we got that the points \bar{e}^1 and \bar{e}^2 exist provided their coordinates satisfy Equations (8) and (15). In the sequel we will show that Equations (8) and (15) are equivalent to $Tr(1) = 1$. Then, $Tr(1) = 1$ if and only if q is an odd power of 2 (see e.g. [9]).

More precisely, we will deal with the cases $s = 1$ and $s \neq 1$ separately showing that

- (i) for $s = 1$ and $s \neq 1$ the choices for (\bar{e}^1, \bar{e}^2) allow exactly two extensions of $C_{\bar{p}}$ to decades;
- (ii) the union of the two decades in (i) is a centric dodecade.

Suppose firstly $s = 1$. Then Equation (15) is

$$e_{1,5}e_{1,6} + e_{1,6}^2(r^2 + \lambda) = 0 \quad (20)$$

and Equation (8) becomes $\lambda = r^2 + r + 1$.

So, $Tr(\lambda) = Tr(r^2 + r + 1) = Tr(1)$. Since $Tr(\lambda) = 1$,

$$Tr(1) = 1. \quad (21)$$

Suppose now $s \neq 1$. Then Equation (15) is

$$e_{1,5}^2 + \frac{e_{1,5}e_{1,6}}{1+s^2} + \frac{(\lambda+r^2)(1+s^2)e_{1,6}^2}{(1+s^2)^2} = 0. \quad (22)$$

Equation (22) admits solutions (in $e_{1,5}$ and $e_{1,6}$) if and only if

$$Tr(\lambda + \lambda s^2 + r^2 + r^2 s^2) = 0. \quad (23)$$

By Equation (8) we have $\lambda s^2 + r^2 + r^2 s^2 = 1 + rs + r^2 s^2$. Hence Equation (23) becomes $Tr(\lambda + \lambda s^2 + r^2 + r^2 s^2) = Tr(\lambda + 1 + rs + r^2 s^2) = 0$ which implies $Tr(\lambda) + Tr(1) + Tr(rs) + Tr(r^2 s^2) = 0$, that is $Tr(1) = 1$, again.

Points \bar{e}^1 and \bar{e}^2 exist if and only if Equation (21) is satisfied, which is equivalent to require $q = 2^n$ with n odd. So, non-linear 5-cliques exist if and only if $q = 2^n$ with n odd. Claim a) is proved.

Let $q = 2^n$ and n odd. Then Equations (20) and (22) admit exactly two different solutions (in $e_{1,5}$ and $e_{1,6}$), up to proportionality: denote them by $(e_{1,5}^*, e_{1,6}^*)$ and $(e_{1,5}^\circ, e_{1,6}^\circ)$. According to (18) and (19) and keeping in mind that the exponents 1 and 2 refer to opposite points, put

$$\bar{e}^{\star 1} := \bar{e}_{(e_{1,5}^*, e_{1,6}^*)}^1, \quad \bar{e}^{\star 2} := \bar{e}_{(e_{1,5}^*, e_{1,6}^*)}^2$$

and

$$\bar{e}^{\circ 1} := \bar{e}_{(e_{1,5}^\circ, e_{1,6}^\circ)}^1, \quad \bar{e}^{\circ 2} := \bar{e}_{(e_{1,5}^\circ, e_{1,6}^\circ)}^2.$$

By construction, $C_{\bar{p}} \cup \{\bar{e}^{*1}, \bar{e}^{*2}\}$ and $C_{\bar{p}} \cup \{\bar{e}^{\circ 1}, \bar{e}^{\circ 2}\}$ are centric decades with center \bar{p} and $C_{\bar{p}} \cup \{\bar{e}^{*1}, \bar{e}^{*2}, \bar{e}^{\circ 1}, \bar{e}^{\circ 2}\}$ is a centric dodecade with center \bar{p} . The Theorem is proved. \square

The following is also implicit in the proof of Theorem 3.14 and corresponds to claims 4.i and 4.ii of Theorem 2.

Corollary 3.15 *Let n be odd. Then every centric cube of $\widehat{\mathcal{X}}$ is contained in exactly two centric decades and just one centric dodecade of $\widehat{\mathcal{X}}$. Every centric decade of $\widehat{\mathcal{X}}$ is contained in exactly one centric dodecade of $\widehat{\mathcal{X}}$.*

Proposition 3.16 *Let $q = 2^n$, n odd. The following hold.*

i) *The number of distinct non-linear 5-cliques of Γ is $N_5 = \frac{(q^4-1)(q^2-1)q^4}{15 \cdot 8}$.*

ii) *The number of distinct non-linear 6-cliques of Γ is $N_6 = \frac{(q^4-1)(q^2-1)q^4}{15 \cdot 48}$.*

Proof. By claim i) of Proposition 3.13, the lifting of a non-linear 5-clique is a centric decade with center n_0 . By Corollary 3.15, the number of decades containing a given centric cube is two. Hence $2 \cdot N_4 = N_5 \cdot \binom{5}{4}$, where $N_4 = \frac{(q^4-1)(q^2-1)q^4}{48}$ is the number of centric cubes with center n_0 by claim b) of Proposition 3.12. Therefore $N_5 = \frac{(q^4-1)(q^2-1)q^4}{15 \cdot 8}$.

The lifting of a non-linear 6-clique is a centric dodecade by claim ii) of Proposition 3.13. By Corollary 3.15, the number of dodecades containing a given centric cube is one. Hence $1 \cdot N_4 = N_6 \cdot \binom{6}{4}$, and therefore $N_6 = \frac{(q^4-1)(q^2-1)q^4}{15 \cdot 48}$. \square

3.4 Proof of Theorem 4.

Claim 1). Let $H = \{a^i, b^i, c^i\}_{i=1,2}$ be a centric hexagon of $\widehat{\mathcal{X}}$ with center n_0 .

The lines $\langle a^1, b^1 \rangle$ and $\langle a^2, b^2 \rangle$ belong to the same plane hence they meet at a point $a^0 \in H_0 \cap Q$, where $H_0 := n_0^\perp$. Similarly, the lines $\langle b^1, c^1 \rangle$ and $\langle b^2, c^2 \rangle$ meet at a point $b^0 \in H_0 \cap Q$ and the lines $\langle a^1, c^2 \rangle$ and $\langle a^2, c^1 \rangle$ meet at the point $c^0 \in H_0 \cap Q$. Let $S := \{a^i, b^i, c^i\}_{i=0,1,2}$ and consider the subgeometry $Q(S)$ of Q induced on S (see Section 1.1, paragraph before Theorem 4, for the definition), namely $Q(S) := (S, \mathcal{L}_S)$ where $\mathcal{L}_S = \{l \cap S : l \text{ is a line of } Q \text{ such that } |l \cap S| \geq 2\}$.

It is straightforward to see that the subgeometry $Q(S)$ is an \mathbb{F}_2 -subgeometry isomorphic to a classical $(2, 1)$ -generalized quadrangle $Q^+(3, 2)$. By construction, the points a^0, b^0, c^0 are orthogonal to n_0 (with respect to the orthogonality relation defined by Q) hence $H \cong Q^+(3, 2) \setminus n_0^\perp$.

Claim 2). Let $C = \{a^i, b^i, c^i, d^i\}_{i=1,2}$ be a cube of $\widehat{\mathcal{X}}$ with center n_0 . Let $\{\hat{l}_1, \hat{m}_1\}, \{\hat{l}_2, \hat{m}_2\}, \dots, \{\hat{l}_6, \hat{m}_6\}$ be the six pairs of opposite edges of C . Then, for $j = 1, \dots, 6$, the lines \hat{l}_j and \hat{m}_j have the same point at infinity $l_j^\infty = m_j^\infty =: p_j$. These six points p_1, p_2, \dots, p_6 are partitioned into three pairs and each of these pairs is collinear in Q_0 with a given point $p_0 \in Q$. Each of these three pairs of

points (on a same line through p_0) corresponds to one of the three classes of parallel edges of the cube. The point p_0 is obtained as the sum of the 4 vertices of any face of the cube.

Let $S := \{a^i, b^i, c^i, d^i\}_{i=1,2} \cup \{p_i\}_{0 \leq i \leq 6}$ and consider the subgeometry $Q(S)$ of Q induced on S .

It is straightforward to see that $Q(S)$ is an \mathbb{F}_2 -subgeometry isomorphic to $Q(4, 2)$. It is also well known from that the complement in $Q(4, 2)$ of a tangent hyperplane is a cube. Hence, $C \cong Q(S) \setminus p_0^\perp$. Note that the projective subgeometry $\langle S \rangle_{\mathbb{F}_2}$ defined by S over the field \mathbb{F}_2 contains the nucleus n_0 of Q_0 but n_0 is not the nucleus n_S of the quadric $Q(S)$ (embedded in $\langle S \rangle_{\mathbb{F}_2}$).

Claim 3) As remarked in Section 1, if $q = 2$ then Γ is a complete graph with 6 vertices. In this case, $Q \setminus Q_0$ is a dodecade.

Let $q \geq 4$. Since by assumption q is an odd power of 2, we can take $\lambda = 1$ in (2); thus Q is represented by the following equation:

$$x_1x_2 + x_3x_4 + x_5^2 + x_5x_6 + x_6^2 = 0. \quad (24)$$

We shall count the number of subgeometries of Q defined over the field \mathbb{F}_2 and represented by the equation (24) for a suitable choice of a basis. Equivalently, we count the number of different bases B_i of the vector space $V = V(6, q)$ with the property that with respect to each of these bases B_i , the quadratic form induced by f on the \mathbb{F}_2 -space spanned by B_i is a quadratic form of elliptic type over \mathbb{F}_2 , hence equivalent to $x_1x_2 + x_3x_4 + x_5^2 + x_5x_6 + x_6^2$.

Let us construct a basis $B = (b_1, b_2, b_3, b_4, b_5, b_6)$ as required above. The first four vectors b_1, b_2, b_3, b_4 can be chosen so that $\langle b_1 \rangle, \langle b_2 \rangle, \langle b_3 \rangle, \langle b_4 \rangle$, is a quadrangle of Q . That is, $f(b_i) = 0$ for $1 \leq i \leq 4$, $b_1 \not\perp b_3$, $b_2 \not\perp b_4$ and $b_1 \perp b_2 \perp b_3 \perp b_4 \perp b_1$.

The number of quadrangles of Q is $(q^3 + 1)(q^2 + 1)(q + 1)q^6$.

The condition $\alpha(b_1, b_2) = 1$ and $\alpha(b_3, b_4) = 1$ (where α stands for the linearization of the form f) still allows us to choose b_1 (or b_2) and b_3 (or b_4) up to arbitrary scalars. Hence, we have $(q^3 + 1)(q^2 + 1)(q + 1)q^6(q - 1)^2$ choices for the quadruple b_1, b_2, b_3, b_4 . The vectors b_5 and b_6 are taken in the orthogonal space $\langle b_1, b_2, b_3, b_4 \rangle^\perp$ of $\langle b_1, b_2, b_3, b_4 \rangle$. (Recall that $\langle b_1, b_2, b_3, b_4 \rangle^\perp$ is a projective line exterior to Q .) So, let us regard b_5 and b_6 as vectors of the 2-dimensional vector space $\langle b_1, b_2, b_3, b_4 \rangle^\perp$.

Let $f'(x, y) = x^2 + \mu xy + \nu y^2$ be the quadratic form induced by f on $\langle b_1, b_2, b_3, b_4 \rangle^\perp$, for a suitable choice of a basis e_1, e_2 of $\langle b_1, b_2, b_3, b_4 \rangle^\perp$. Hence $Tr(\nu/\mu^2) = 1$ because the line $\langle b_1, b_2, b_3, b_4 \rangle^\perp$ is external to Q . Put $b_5 := x_1e_1 + x_2e_2$ and $b_6 := y_1e_1 + y_2e_2$ for scalars x_1, x_2, y_1, y_2 to be determined in a few lines.

The conditions to require on b_5 and b_6 are the following: $\alpha(b_5, b_6) = 1$, $f(b_5) = 1$ and $f(b_6) = 1$. By Lemma 3.5, we have $q + 1$ different choices for b_5 . Indeed, $f(b_5) = 1$ can be translated to $x_1^2 + x_1x_2 + x_2^2 = 1$.

Having chosen b_5 , let us count how many choices remain for b_6 .

To fix ideas put $b_5 = e_1$. Then $\alpha(b_5, b_6) = 1$ implies $y_2 = 1/\mu$ and $f(b_6) = 1$ implies $y_1^2 + \mu y_1 y_2 + \nu y_2^2 = 1$, namely $y_1^2 + y_1 + \nu/(\mu^2) + 1 = 0$. Since the equation

$y_1^2 + y_1 + \nu/(\mu^2) + 1 = 0$ has two distinct solutions, two different choices are left for b_6 . A similar argument works with any other choice of b_5 . Thus, in any case, only two choices are left for b_6 once b_5 has been chosen. So, the number of \mathbb{F}_2 -subgeometries $Q(S)$ of $Q^-(5, q)$ isomorphic to $Q^-(5, 2)$ is

$$\begin{aligned} & \frac{\# \text{ choices of } (b_i)_{i=1}^6 \text{ in } V(6, q)}{\# \text{ choices of } (b_i)_{i=1}^6 \text{ in } V(6, 2)} = \\ & \frac{(q^3 + 1)(q^2 + 1)(q + 1)q^6(q - 1)^2(q + 1)2}{(2^3 + 1)(2^2 + 1)(2 + 1)2^6(2 - 1)^2(2 + 1)2} = \\ & \frac{(q^3 + 1)(q^2 + 1)(q + 1)^2q^6(q - 1)^2}{9 \cdot 5 \cdot 3^2 \cdot 2^6}. \end{aligned} \quad (25)$$

By Proposition 3.16, the total number \bar{N}_6 of centric dodecades of Q (any possible center being allowed) is

$$\begin{aligned} & (\# \text{ of dodecades with a given center}) \cdot |\text{PG}(V) \setminus Q| = \\ & N_6 \cdot |\text{PG}(V) \setminus Q| = \frac{(q^4 - 1)(q^2 - 1)q^4}{15 \cdot 48} \cdot \left[\frac{(q^6 - 1)}{q - 1} - (q^3 + 1)(q + 1) \right] = \\ & \frac{(q^3 + 1)(q^4 - 1)(q^2 - 1)q^6}{48 \cdot 15}. \end{aligned}$$

Hence $\bar{N}_6 = \frac{(q^3 + 1)(q^4 - 1)(q^2 - 1)q^6}{48 \cdot 15}$.

Note that if $Q(S)$ is an \mathbb{F}_2 -subgeometry of Q and $p \in \langle S \rangle_{\mathbb{F}_2} \setminus S$, then $Q(S) \setminus p^\perp$ is a dodecade with center p . For each such subgeometry $Q(S)$ there are $2^2(2^3 + 1) = 4 \cdot 9$ possible choices for p . In view of (25) this accounts for a total of $\frac{(q^3 + 1)(q^2 + 1)(q + 1)^2q^6(q - 1)^2}{9 \cdot 5 \cdot 3^2 \cdot 2^6} \cdot 4 \cdot 9 = \frac{(q^3 + 1)(q^4 - 1)(q^2 - 1)q^6}{5 \cdot 9 \cdot 2^4} = \bar{N}_6$ centric dodecades.

Hence every dodecade with center p corresponds to a subgeometry $Q(S) \setminus p^\perp$ over \mathbb{F}_2 as described above and viceversa. \square

Acknowledgements. The authors wish to thank Antonio Pasini for his very helpful remarks and comments on a first version of this paper.

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